Lecture 21 (2. June. 2023)

Review of section 7

The purpose of sect. 7 was to motivate the study of non-holomorphic modular functions by showing their application to the theory of quadratic forms and class numbers

① Binary quadratic forms (integral, primitive)

An integral, primitive, binary quad. form is

$$Q(x,y) = Ax^2 + Bxy + Cy^2 \in \mathbb{Z}[x,y]$$

with $\gcd(A,B,C) = 1$

Notation: $Q = [A,B,C]$

Def: The discriminant of $Q = [A,B,C]$ is $D := B^2 - 4AC$

Note that $D \equiv B^2 \equiv 0,1 \mod 4$

Exercise: $[A,B,C]$ is irred. in $\mathbb{Z}[X]$ $\Leftrightarrow$ D is not a perfect square (in $\mathbb{Z}$) (assuming $\gcd(A,B,C) = 1$)

We assume D not a perfect square in what follows ("$D \neq \square$")

If $D < 0$ $\leadsto$ $A > 0$: Q positive definite, ie $Q(x,y) > 0$ $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

or

$A < 0$: Q negative definite, ie $Q(x,y) < 0$ $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

Since we can change $[A,B,C]$ to $[-A,-B,-C]$, we assume $A > 0$

If $D > 0$ $\leadsto$ Q is indefinite, ie has positive and negative values

Given D disc. ($D \neq \square$, $D \equiv 0,1 \bmod 4$) define

$$\mathcal{Q}_D = \left\{ [A,B,C] \text{ integral, primitive, binary quad. forms of disc } B^2-4AC = D \text{ (with } A>0 \text{ if } D<0) \right\}$$

$\Gamma = SL_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ Q(X,Y) = Q\left((X,Y)\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t\right) = Q(aX+bY,\ cX+dY)$$

(this is a left action)

Matrix representation

$Q = [A,B,C] \leftrightarrow \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} =: M_Q$ $\qquad Q(X,Y) = (X\ Y)\underbrace{\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}}_{M_Q}\begin{pmatrix} X \\ Y \end{pmatrix}$

If $\gamma \in SL_2(\mathbb{Z})$ then $M_{\gamma Q} = \gamma^t M_Q \gamma$

Note that $\operatorname{disc}(Q) = B^2-4AC = -4\cdot\det(M_Q)$. In particular $\operatorname{disc}(\gamma\cdot Q) = \operatorname{disc}(Q)$.

Theorem 7.1 (Lecture 16): $\Gamma\backslash\mathcal{Q}_D$ is finite

$h(D) := \#(\Gamma\backslash\mathcal{Q}_D)$ is called the class number of disc. D

When $D<0$ a set of reps for $\Gamma\backslash\mathcal{Q}_D$ can be found using the following:

Theorem 7.3 (Lecture 17) Assume $D<0$. Any form in $\mathcal{Q}_D$ is Γ-equivalent to a unique form $[A,B,C]$ which satisfies

(*) $|B| \leq A \leq C$, and $B \geq 0$ if either $|B| = A$ or $A = C$

If $[A,B,C]$ satisfies (*) then we call it "reduced"

The map $\mathcal{Q}_D \longrightarrow \mathbb{H}$ $(D<0)$

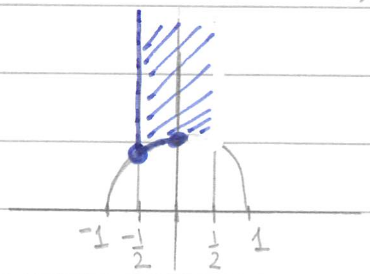
Define $\mathcal{Q}_D \longrightarrow \mathbb{H}$

$Q \longmapsto \tau_Q$ = unique point in $\mathbb{H}$ solution of $Q(x,1)=0$

If $Q=[A,B,C]$ then $\tau_Q = \dfrac{-B+i\sqrt{|D|}}{2A}$

Ex: If $\gamma\in\Gamma$ then $\tau_{\gamma Q} = \gamma^{-1}\cdot\tau_Q$

Prop 7.3 (Lecture 17) A form $Q\in\mathcal{Q}_D$ $(D<0)$ is reduced iff $\tau_Q \in \left\{ z\in\mathbb{C} : |Re(z)|\leq 1/2,\ |z|\geq 1 \text{ and } Re(z)\leq 0 \text{ if } |Re(z)|=1/2 \text{ or } |z|=1 \right\}$



Representation of integers by quadratic forms

A quadratic form Q represents $n\in\mathbb{Z}$ if $\exists (x,y)\in\mathbb{Z}^2$ such that $Q(x,y)=n$. The representation is primitive if $\gcd(x,y)=1$.
The group $\Gamma_Q := \{\gamma\in\Gamma = SL_2(\mathbb{Z}) : \gamma Q = Q\}$ acts on representations of n and also on primitive representations

$\rightsquigarrow$ $r_Q(n) := \#\left(\Gamma_Q \backslash \{(x,y)\in\mathbb{Z}^2 \text{ primitive},\ Q(x,y)=n\} \right)$

Rmk: ① $\#\left(\Gamma_Q\backslash\{(x,y)\in\mathbb{Z}^2,\ Q(x,y)=n\}\right) = \displaystyle\sum_{\substack{m/n \\ \frac{n}{m}=\square}} r_Q(m)$

② $r_{\gamma Q}(n) = r_Q(n)$ $\forall\gamma\in\Gamma, \forall n\in\mathbb{Z}$ so $r_Q(n)$ if well defined for $[Q]\in\Gamma\backslash\mathcal{Q}_D$

How large is Γ_Q?

Theorem 7.5 (Lecture 18) For $Q=[A,B,C]\in\mathcal{Q}_D$ we have bijection

$$\{(t,u)\in\mathbb{Z}^2 : t^2-Du^2=4\} \longrightarrow \Gamma_Q$$

$$(t,u) \longmapsto \begin{pmatrix} \frac{t-Bu}{2} & -Cu \\ Au & \frac{t+Bu}{2} \end{pmatrix}$$

Moreover: If $D>0$ then $\Gamma_Q \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}$

If $D<0$ then $\#\Gamma_Q = \begin{cases} 6 & \text{if } D=-3 \\ 4 & \text{if } D=-4 \\ 2 & \text{if } D<-4 \end{cases}$

For $D<0$ we have

$$r_Q(n) = \frac{1}{\omega_D}\#\{(x,y)\in\mathbb{Z}^2 \text{ primitive}, Q(x,y)=n\} \text{ with}$$

$$\omega_D = \begin{cases} 6 & \text{if } D=-3 \\ 4 & \text{if } D=-4 \\ 2 & \text{if } D<-4 \end{cases}$$

Def: Given D disc define $r_D(n) = \sum_{[Q]\in\Gamma\backslash\mathcal{Q}_D} r_Q(n)$

Theorem 7.7 (Lecture 18) For $D<0$ disc and $n\geq 1$ integer

$$r_D(n) = \#\{b \bmod 2n \mid b^2\equiv D \bmod 4n\}$$

(2) Kronecker symbol

To connect quadratic forms with analytic number theory we need to introduce the Kronecker symbol $\left(\frac{D}{\cdot}\right)$ for D fundamental discriminant

Def: A discriminant $D\in\mathbb{Z}$ ($D\neq\square$, $D\equiv 0,1 \bmod 4$) is called fundamental if either $D\equiv 1 \bmod 4$ and D is square-free

or $D\equiv 0 \bmod 4$ and $\frac{D}{4}\equiv 2$ or $3 \bmod 4$ with $\frac{D}{4}$ square-free

Rmk: Fundamental discriminants are exactly discriminants of quadratic extensions of $\mathbb{Q}$

(Any $K/\mathbb{Q}$ quad. is $K = \mathbb{Q}(\sqrt{d})$ $d \in \mathbb{Z}$ square-free with $d \neq 1$
ring of integers $\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & \text{if } d \equiv 2,3 \ (4) \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & \text{if } d \equiv 1 \ (4) \end{cases}$

hence $D := \operatorname{disc}(K) := \det \begin{pmatrix} 1 & \sqrt{d} \\ 1 & -\sqrt{d} \end{pmatrix}^2$ or $\det \begin{pmatrix} 1 & \frac{1+\sqrt{d}}{2} \\ 1 & \frac{1-\sqrt{d}}{2} \end{pmatrix}^2$

$= 4d$ or d)

Def Given D fund discriminant define $\chi_D : \mathbb{Z}^+ \longrightarrow \{-1, 0, 1\}$
as $\chi_D(p) = \left(\frac{D}{p}\right)$ if p odd prime

$$\chi_D(2) = \begin{cases} 0 & \text{if } D \equiv 0 \ (4) \\ 1 & \text{if } D \equiv 1 \ (8) \\ -1 & \text{if } D \equiv 5 \ (8) \end{cases}$$

and extend χ_D to $\mathbb{Z}^+$ as a completely multiplicative function
This is the Kronecker symbol of the discriminant D

Theorem 7.8 (Lecture 19) For $D < 0$ fundamental and $n \in \mathbb{Z}^+$

$$r_D(n) = \begin{cases} 0 & \text{if } p^2 / n \text{ for some prime } p / D \\ \prod_{p/n} (1 + \chi_D(p)) & \text{otherwise} \end{cases}$$

Analytic formulation of Theorem 7.8: For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$

$$\zeta(2s) \cdot \sum_{n=1}^{\infty} \frac{r_D(n)}{n^s} = \zeta(s) \cdot L(\chi_D, s)$$

where $L(\chi_D, s) = \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n^s}$ Dirichlet L-function
associated to χ_D (χ_D is a Dirichlet character mod $|D|$
i.e. it is the lift to $\mathbb{Z}$ of $(\mathbb{Z}/D\mathbb{Z})^\times \longrightarrow \mathbb{C}$ gp. homomorphism)

(3) Non-holomorphic Eisenstein series and Epstein zeta functions

For $\tau \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ define (Lecture 19)

$$G(z,s) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{y^s}{|mz+n|^{2s}} = \frac{\zeta(2s)}{2} \sum_{\substack{(c,d) \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$$

Then $G(z,s)$ is Γ-invariant in z and has meromorphic continuation to $s \in \mathbb{C}$

Lemma (Lecture 19) For $D<0$ disc, $[Q] \in \Gamma \backslash \mathcal{Q}_D$ (so $[\tau_Q] \in \Gamma \backslash \mathbb{H}$) we have

$$G(\tau_Q, s) = \left(\frac{|D|}{4}\right)^{s/2} Z_Q(s)$$

where

$$Z_Q(s) := \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{Q(m,n)^s} = \frac{\omega_D}{2} \cdot \zeta(2s) \sum_{n=1}^{\infty} \frac{r_Q(n)}{n^s}$$

is the Epstein zeta function associated to Q (or $[Q]$)

Def: $$Z_D(s) = \frac{2}{\omega_D} \sum_{[Q] \in \Gamma \backslash \mathcal{Q}_D} Z_Q(s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{r_D(n)}{n^s} = \zeta(s) L(\chi_D, s)$$

From Exercise Sheet #6 we know

$$E^*(z,s) := \pi^{-s} \Gamma(s) \sum_{\substack{(m,n) \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{y^s}{|mz+n|^{2s}} = \pi^{-s}\Gamma(s) \cdot 2G(z,s)$$

has simple poles at $s=0,1$ with residues -1 and 1

$$\rightsquigarrow \operatorname{Res}_{s=1} G(z,s) = \frac{\pi}{2} \quad \rightsquigarrow \operatorname{Res}_{s=1} Z_Q(s) = \frac{\pi}{\sqrt{|D|}}$$

$$\rightsquigarrow \operatorname{Res}_{s=1} Z_D(s) = \frac{2}{\omega_D} \frac{\pi}{\sqrt{|D|}} h(D)$$

$\operatorname{Res}_{s=1} Z_D(s) = L(\chi_D, 1)$ if $D<0$ fundamental

This proves:

Theorem (Analytic class number formula) For $D<0$ fundamental

$$L(\chi_D, 1) = \frac{2\pi\, h(D)}{w_D \cdot \sqrt{|D|}}$$

where $L(\chi_D,1) = \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n} = \prod_p \left(1-\chi_D(p)p^{-1}\right)^{-1}$

(conditionally convergent)

Rmk: For $D>0$ one can prove $L(\chi_D,1) = \frac{2h(D)\log(\varepsilon_0)}{\sqrt{D}}$

where $\varepsilon_0 := \frac{t_0 + u_0\sqrt{D}}{2}$ and (t_0,u_0)

is the smallest positive solution of $t^2 - Du^2 = 4$.

4 Maass forms (Hans Maass, 1911–1992)

$G(z,s)$ is an example of Maass form for $\Gamma = SL_2(\mathbb{Z})$

It is not holomorphic in z but satisfies

$$\Delta G(z,s) = s(s-1)\, G(z,s)$$

where $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, $z = x+iy$ hyperbolic Laplacian (of weight 0)

so $G(z,s)$ is eigenvector of Δ with eigenvalue $s(s-1)$ (for $s \in \mathbb{C}\setminus\{0,1\}$ fixed)

Fact: For any smooth $f:\mathbb{H}\to\mathbb{C}$ any $\gamma\in\Gamma$: $\Delta(f|_0\gamma) = \Delta(f)|_0\gamma$

where $(f|_0\gamma)z := f(\gamma z)$

Def: A Maass form for Γ (of weight 0) is a smooth function $f:\mathbb{H}\to\mathbb{C}$ satisfying

i) f is Γ-inv

ii) $\Delta f = \lambda f$ for some $\lambda\in\mathbb{C}$ ($\leadsto \lambda\in\mathbb{R}$)

iii) $f(z) = O(y^A)$ for some $A>0$ when $\mathrm{Im}(z)\to\infty$

Fact: The $\mathbb{C}$-vector space $M_{0,\lambda}(\Gamma) = \{$Maass forms wt. 0, Δ-eigenvalue $=\lambda\}$ is of finite dimension.

Ex: $G(z,s) \in M_{0,s(1-s)}(\Gamma)$

<u>Theorem</u>: Any $f \in M_{0,\lambda}(\Gamma)$ with $\lambda = s(1-s)$ has Fourier expansion of the form

$$f(z) = a_0^+ y^s + a_0^- y^{1-s} + \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n W_s(nz) \qquad (*)$$

where $W_s(nz) := 2\sqrt{y}\, K_{s-\frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}$

and $K_\nu(y) := \frac{1}{2} \int_0^\infty e^{-y(t+t^{-1})/2} t^\nu \frac{dt}{t}$ (K-Bessel function)

<u>Def</u>: $f \in M_{0,\lambda}(\Gamma)$ is a cusp form if $a_0^+ = a_0^- = 0$ in $(*)$

One can define Hecke operators and show that the spaces of Maass cusp forms have bases formed by simultaneous eigenfunctions (Hecke-Maass forms), but still the Fourier coeffs a_n ($n \neq 0$) are very mysterious